MATH 579: Combinatorics Final Exam

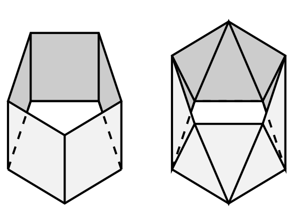
Please read the following instructions. For the following exam you may not use any papers, books, or computers, apart from a single $3'' \times 5''$ card, and a calculator. Please turn in **exactly ten** problems. You must do problems 1-6, and four more chosen from 7-12. Please write your answers on separate paper, make clear what work goes with which problem, and put your name or initials on every page. You have 120 minutes. Be sure to adequately justify all your solutions. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 50 and 100.

Turn in problems 1-6:

1. Compute the number of subsets of [20] that contain exactly five elements from [10].
2. Use difference calculus to compute $\sum_{i=0}^{20} i^3 - i^2$.
3. Compute the number of integers in [1000] relatively prime to 42.
4. Consider a pentagonal prism, with the pentagonal faces labelled as $6, 7$, and the square faces labeled as $1, 2, 3, 4, 5$ in turn. Give explicitly the subgroup of S_7 corresponding to rotations of the prism.
5. Compute the number of ways to color each face of a pentagonal prism either black or white, up to rotation.
6. Compute the number of ways to color each face of a pentagonal antiprism either black or white, up to rotation.

Turn in exactly four more problems of your choice:

7. Count the number of ordered pairs (A, B), such that A, B are each subsets of [10], and satisfy $A \cap B \neq \emptyset$.
8. Prove that, for all $n \in \mathbb{N}$, $p(n)^2 < p(n^2 + 2n)$.
9. Prove the Hexagon Identity: for all integers m, n with $n > m > 0$, $\binom{n-1}{m}\binom{n}{m-1}\binom{n+1}{m+1} = \binom{n-1}{m-1}\binom{n}{m+1}\binom{n+1}{m}$.
10. Recall the Fibonacci sequence, given by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$. Solve the recurrence given by $a_0 = 2, a_1 = 1 - 2\sqrt{3}, a_n = 2a_{n-1} + 2a_{n-2} + F_n$ $(n \geq 2)$.
11. Let b_n denote the number of subsets of $[n]$, such that the difference between any two elements within a subset is at least three. Find the generating function for $\{b_n\}$.
12. Compute the number of ways to color each edge of a pentagonal antiprism either black or white, up to all symmetries of the antiprism.



The left figure is a pentagonal prism. Its faces consist of two regular pentagons (pictured as transparent), and five squares (pictured as shaded). The right figure is a pentagonal antiprism. Its faces consist of two regular pentagons (transparent), and ten equilateral triangles (shaded).